hep-th/0505054
UCI-TR-2005-15
UCSD-PTH-05-03
MCTP-05-74
May 2005

High-Energy Scattering in Non-Commutative Field Theory

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Abstract

We analyze high energy scattering for non-commutative field theories using the dual gravity description. We find that the Froissart-Martin bound still holds, but that cross-sections stretch in the non-commutative directions in a way dependent on the infrared cutoff. This puzzling behavior suggests new aspects of UV/IR mixing.

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1 Introduction

In standard quantum field theory, the cross-section for a process cannot grow arbitrarily quickly at high energies. This is formalized by the Froissart-Martin bound [1, 2], which states that the cross-section can only increase with energy at most as

$$\sigma = C \log^2 s \tag{1}$$

where C is constant.

This bound is only relevant for theories with no massless particles. In particular, quantum gravity appears to violate this bound. At high energies, the cross-section is dominated by black hole production [3, 4, 5, 6] and since the radius of the black hole increases with the energy, one expects the cross-section to show power-law increase as one goes to the UV.

These competing intuitions come into conflict in the case of non-commutative field theory (NCFT). Although they are field theories, they display many properties of quantum gravity in general and string theory in particular [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. Since a NCFT contains interactions of arbitrarily high dimension, the standard field-theoretic arguments for the Froissart bound do not necessarily apply. And although a NCFT can contain a naive mass gap, UV/IR mixing [8, 9] implies that poles at zero momentum may nevertheless arise due to UV effects. As a result, it seems quite possible that NCFTs, even with a naive mass gap, may nevertheless violate the Froissart bound and display a power-law growth more consistent with quantum gravity intution.

In quantum gravity, general arguments indicate that high-energy scattering is dominated by black hole production. One might similarly expect that non-perturbative contributions to scattering in NCFT become dominant at high energies. Non-perturbative contributions are expected to scale as $e^{\frac{1}{g^2}}$, where g is a dimensionless coupling of the theory. In NCFT there are dimensionless couplings of the form θE^2, where θ is the non-commutativity parameter and E is the characteristic energy of the process. Since this theory does not break down at any scale, it should be valid up to energies at which $\theta E^2 \sim 1$, when non-perturbative corrections should become important to scattering. And indeed, NCFT contains non-perturbative solitons [23] [24] [25] whose size scale with area which would be just right to play the role of the black hole in scattering. Furthermore, there is a stretched-string effect

in NCFT [22] whereby the size of bifundamental particles increases in the non-commutative directions.

All of this makes it plausible that high-energy scattering in NCFT is dominated by non-commutative solitions, in analogy to the black holes of quantum gravity, but in contrast to our usual notions of field theories. Our goal in this paper will be to use the AdS/CFT correspondence (in a manner analagous to [28] in the commutative context) to test this intuition. This should provide insight into the nature of the deep relation between NCFT and quantum gravity.

In section 2, we review the calculation, due to Giddings [28], of the Froissart bound in commutative field theory from AdS/CFT. In section 3 we use an analogous calculation to determine the scaling of cross-sections with energy in non-commutative field theory. In section 4, we find that cross-sections stretch in the non-commutative directions, and in section 5 we conclude with a discussion of how these results impact our understanding of UV/IR mixing.

2 The Froissart Bound from AdS/CFT

It was shown by Giddings [28] (and further elucidated by Kang and Nastase [31, 32]) that the Froissart-Martin bound for a gauge theory could be derived using the AdS/CFT correspondence. As is well-known, string theory in an $AdS_5 \times S^5$ background is dual to $N = 4$ gauge theory on the boundary of spacetime. According to AdS/CFT, scattering amplitudes in the boundary theory correspond to scattering amplitudes in the bulk theory. The corresponding cross-section for the bulk scattering process, when projected onto the brane, yields the cross-section for the gauge theory process [33][34]. In fact, high energy bulk gravitational scattering is dominated by black hole production, and the corresponding behavior of the bulk cross-section precisely corresponds to a process which saturates the Froissart-Martin bound in the dual boundary theory.

In the boundary gauge theory, the momentum is

$$p_a = -\imath \frac{\partial}{\partial x^a} \tag{2}$$

This momentum is a conserved quantity, as it generates an isometry (translation by x^a). But an observer in the bulk will measure a momentum

$$\tilde{p}_\mu = e_\mu{}^a p_a \tag{3}$$

in a local inertial frame. Bulk amplitudes will depend on this momentum evaluated in the bulk at the point of interaction. An incoming boundary scattering state will appear in the bulk scattering process as an incoming wavefunction of the form

$$\psi \sim e^{ip\cdot x} f(u) \tag{4}$$

where we ignore the dependence on the transverse S^5. The bulk and boundary scattering amplitudes are then related by

$$A_{gauge}(p) = \int d^5x\sqrt{-g} A_{bulk}(x^\mu, u)\Pi_i \psi_i \tag{5}$$

We can find the bulk cross section by assuming that the high energy behavior is dominated by black hole formation. The mass of the black hole will be the bulk energy of the collision. The cross-section for bulk scattering is then proportional to the square of the black hole radius. Having computed the bulk cross-sectional radius $\tilde{r}$, we can determine the scale r of the corresponding gauge theory process by inverting the relation:

$$\tilde{r}_\mu = e_\mu^{\ a} r_a \tag{6}$$

We can now derive the Froissart bound for ordinary gauge theories, following [28]. We start with the $AdS_5 \times S^5$ metric

$$\begin{aligned} ds^2 &= \alpha' R^2 \left[u^2(-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{du^2}{u^2} + d\Omega_5^2 \right] \\ &= \alpha' \left[dy^2 + e^{-2y/R} dx^\mu dx_\mu + R^2 d\Omega_5^2 \right] \end{aligned} \tag{7}$$

where μ runs from 0 to 3, and $u = \frac{1}{R} e^{-\frac{y}{R}}$. The boundary is at $\frac{1}{u} = z = 0$.

The interpretation of scattering in pure $AdS_5 \times S^5$ is problematic because the maximally supersymmetric gauge theory is conformal. This can be rectified by considering a deformation of the gauge theory which makes it non-conformal. Correspondingly, the dual geometry will be modified in the infrared.

This deformation can be done in many ways; for instance one could consider the $N = 1^*$ theory of [26], or the finite temperature theory [27]. Since the high-energy behavior is expected to be universal, we will simply truncate the geometry by putting an infrared cutoff brane at $u = u_c$ [28]. Without loss of generality, we can take $u_c = R^{-1}$.

The high energy behavior of scattering processes is dominated by the production of black holes in this geometry. Unfortunately, the exact metric of these black holes is unknown. We can however estimate the size of the black holes by using a procedure suggested by Giddings. In this procedure, one solves for the linearized perturbations produced by a mass m in this geometry. The location where the perturbations become of order 1 should approximately reproduce the horizon geometry.

For this purpose, we can rewrite the metric as

$$ds^2 = (1 + h_{yy})dy^2 + e^{-2y/R}(\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu \tag{8}$$

where h is the perturbation of the metric (we are ignoring any dependence on the S^5, since we are looking for spherically symmetric solutions.)

In an appropriate gauge, the linearized metric fluctuations will satisfy the bulk equation [28][29]

$$D^2 h_{\mu\nu} = 0 \tag{9}$$

We need to supplement this with boundary conditions in the UV and IR. In the UV ($z \sim 0$), the condition is that the field needs to fall off sufficiently rapidly. In the IR, we cutoff the space with a brane which has some fixed energy density $T_{\mu\nu} = S_{\mu\nu}\delta(y)$. The boundary conditions for fields at this brane may be determined by integrating the equations of motion over a small region around the brane. We consider the approximation in which the brane is rigid (the so-called "heavy radion" limit) and find the boundary condition

$$\partial_y(h_{\mu\nu} - \eta_{\mu\nu}h)|_{y=0} - \frac{d-1}{R} h_{yy}(0)\eta_{\mu\nu} = \frac{S_{\mu\nu}}{2M_p^{d-1}} \tag{10}$$

where $h = \eta^{\mu\nu}h_{\mu\nu}$, $u = \frac{1}{R}e^{-\frac{y}{R}}$ and $d = 4$. The linearized equation of motion for h then is solved by:

$$h_{\mu\nu} = \frac{1}{2M_p^{d-1}} \int \frac{d^dx'}{(2\pi)^d}\sqrt{g}\Delta_{d+1}(X, 0; x') \times \left[S_{\mu\nu}(x') - \eta_{\mu\nu}\frac{S_\rho{}^\rho(x')}{d-1} + \frac{\partial_\mu\partial_\nu}{\partial^2}\frac{S_\rho{}^\rho(x')}{d-1}\right] \tag{11}$$

where the function Δ satisfies

$$D^2\Delta_{d+1}(X, 0; x') = \frac{\delta^{d+1}(X - X')}{\sqrt{G}} \tag{12}$$

$$\partial_y\Delta_{d+1}(X, X')|_{y=0} = 0 \tag{13}$$

Δ is thus the Neumann Green's function for the scalar Laplacian

$$\Delta_{d+1}(x,u,x',R) = -\left(\frac{1}{uR}\right)^{\frac{d}{2}} \int \frac{d^d p}{(2\pi)^d} \frac{1}{q} \frac{J_{\frac{d}{2}}(\frac{q}{u})}{J_{\frac{d}{2}-1}(qR)} e^{ip(x-x')} \tag{14}$$

with $q^2 = -p^2$.

If we place a mass m at the point $u^{-1} = R, x' = 0$, we will produce perturbations around the background metric. We then estimate the horizon of the black hole as the place where the perturbations are $\mathcal{O}(1)$:

$$1 \sim \frac{m}{2M_p^{d-1}} \sqrt{g} \Delta_{d+1}(X, 0; 0) \tag{15}$$

One can compute the p integral by contour integration, and the integral will pick up the poles of the integrand at the zeros of the denominator. We are only interested in the limit of large m (where m is the mass of the black hole) and this limit is dominated by $q \sim 0$, leaving

$$\Delta_{d+1}(x,u,x',R) = \int \frac{d^d p}{(2\pi)^d} \frac{1}{q} \frac{(\frac{qR}{2})^{\frac{d}{2}}}{J_{\frac{d}{2}-1}(qR)} \frac{1}{\Gamma(\frac{d}{2}+1)} e^{ip(x-x')} \tag{16}$$

The integral is dominated by the first zero of the denominator, $q = M_1$. Evaluating the residue at this pole and performing the angular integrations, one finds that the linearized perturbation satisfies

$$h_{00} \quad \propto \quad m u^{-4} \frac{e^{-M_1 r}}{r} \tag{17}$$

where r is the distance in the longitudinal directions, the UV boundary is at $u = \infty$ $(y = -\infty)$ and the IR brane is at $\frac{1}{u} = R$ $(y = 0)$. The pole in Δ_p is located at $p = \imath M_1$. Setting $h_{00} \sim 1$ and taking the logarithm of both sides gives

$$M_1 r + \log(M_1 r) \sim \log(m M_1) + 4y \tag{18}$$

Near $r = 0$ the solution is expected to be smoothed out [28]. Since the warp factor must be monotonic, the horizon size is maximized at the IR brane $(y = 0)$ and decreases as y decreases (i.e. as one moves to the UV). Thus the longitudinal size of the horizon is maximized when

$$r \sim \frac{1}{M_1} \log(\frac{m}{M_1}) \tag{19}$$

This corresponds to a cross section $\sigma = \pi r^2 \sim (\ln s)^2$. The cross-section hence saturates the Froissart bound.

3 The Noncommutative Case

We will now use similar arguments to find the Froissart bound for the case of noncommutative field theory. The gravity solution dual to an NCFT was found in [29] and [30]. This background is described by the solution

$$ds^2 = \alpha' R^2 \left[u^2(-dx_0^2 + dx_1^2) + \frac{u^2}{1+a^4u^4}(dx_2^2 + dx_3^2)) + \frac{du^2}{u^2} + d\Omega_5^2 \right]$$

$$B_{23} = \alpha' R^2 \frac{a^2u^4}{1+a^4u^4} \qquad e^{2\varphi} = \frac{g^2}{1+a^4u^4}$$

$$A_{01} = \alpha' \frac{b}{g} u^4 R^4 \qquad F_{0123u} = \frac{\alpha'^2}{g(1+a^4u^4)} \partial_u (u^4 R^4) \tag{20}$$

The boundary of the space is $u = \infty$. For small u, the solution becomes $AdS_5 \times S^5$ (with the relation $u = \frac{1}{z}$).

We shall now put a cutoff brane at $u = u_c$. This will correspond to the cutoff brane considered by [28]. Scattering processes at high energy will then correspond in the dual theory to the formation of a black hole of mass M at the point $u = u_c$. To find the approximate shape of the black hole, we shall consider the linearized response to a source placed at the origin of x_i and at $u = u_c$. The horizon will be approximately at the place where the fluctuations become large and the linearized approximation breaks down.

The calculation in this background will be a good deal more complicated than the calculation in $AdS^5 \times S^5$, because the background geometry is less symmetric and involves more fields. Perturbations of the metric will now be coupled to perturbations of the gauge fields. Furthermore, in this case, unlike AdS^5, there is a new scale a in the problem. The shape of the black hole will now depend on the relative sizes of u_c and a. To avoid these technical complications, we shall instead consider as a model the fluctuations of a scalar field with the action

$$S = \int d^{10}x \sqrt{g} e^{-2\varphi} (\partial\phi)^2 \tag{21}$$

and Neumann boundary conditions. (Here φ is the background dilaton, and ϕ is a different field.)

The equation of motion for the field is

$$\partial_\mu (\sqrt{g} g^{\mu\nu} e^{-2\varphi} \partial_\nu \phi) = 0 \tag{22}$$

For linearized fluctuations around the geometry (20), this reduces to

$$\frac{1}{u^5}\partial_u(u^5\partial_u\phi) + \frac{1}{u^4}(-\partial_0^2 + \partial_1^2)\phi + \frac{1}{u^4}(1 + a^4u^4)(\partial_2^2 + \partial_3^2)\phi = 0 \tag{23}$$

We shall further assume that the scalar is coupled to the external source mass M by a coupling

$$S = \int d^{10}x\sqrt{g}Me^{-2\varphi}\phi \tag{24}$$

Then in the presence of a mass M located at $u = u_M, x_1 = x_2 = x_3 = 0$, the equation of motion is modified to be

$$\frac{1}{u^5}\partial_u(u^5\partial_u\phi) - \frac{1}{u^4}k_1^2\phi - \frac{1}{u^4}(1 + a^4u^4)(k_2^2 + k_3^2)\phi = M\frac{1}{u^5}\delta(u - u_M) \tag{25}$$

where we have Fourier transformed ϕ by

$$\phi(x, u) = \int d^3k\ e^{ik_1x^1+ik_2x^2+ik_3x^3}\phi(k_1, k_2, k_3; u) \tag{26}$$

We will initially place the black hole source in the bulk of space-time, and then take the limit as it approaches the IR cutoff brane (i.e. $u_M \to u_c$). We will designate the solution for ϕ in the region between the black hole and the IR cutoff as ϕ_-, and the solution for ϕ in the region between the black hole and the UV cutoff as ϕ_+,

Regularity of the solution mandates that ϕ_+ should fall off as $u \to \infty$. On the other hand, ϕ_- satisfies a Neumann boundary condition at the brane $\partial_u\phi_- = 0$ at $u = u_c$. ϕ_+ and ϕ_- must be matched by integrating (25) across the source.

In the limit $u_M \to u_c$ we can focus on ϕ_+. The entire solution is then determined to be

$$\phi(x, u) = \frac{M}{u_c^5}\int d^3k\ e^{ik_1x^1+ik_2x^2+ik_3x^3}\frac{\phi_+(k_1, k_2, k_3, a; u)}{\partial_u\phi_+(k_1, k_2, k_3, a; u = u_c)} \tag{27}$$

When we look for the limiting cross-section, we want to maximize the black hole area; that is, we want to know the largest radius where ϕ becomes of order 1. This must occur at $u = u_M$, due to the monotonicity of the warp factor. Hence we must solve the equation

$$\frac{M}{u_c^5}\int d^3k\ e^{ik_1x^1+ik_2x^2+ik_3x^3}\frac{\phi_+(k_1, k_2, k_3, a; u_c)}{\partial_u\phi_+(k_1, k_2, k_3, a; u_c)} = 1 \tag{28}$$

To get an idea of the shape of the black hole, we will look for an solution for ϕ_+. The exact solution is unwieldly and not very helpful, so we instead will look for an approximate solution. We shall do this by solving the equation (25) for $u \gg \frac{1}{a}$ and $u \ll \frac{1}{a}$ separately, and matching them at $u \sim a$.

First, we consider the case $u \gg \frac{1}{a}$. Here the equation for ϕ_+ simplifies to

$$\frac{1}{u^5}\partial_u(u^5\partial_u\phi) - a^4(k_2^2 + k_3^2)\phi = 0 \tag{29}$$

with the solution

$$\phi = u^{-2}\left(C_1 K_\nu(lu) + C_2 I_\nu(lu)\right) \tag{30}$$

where $\nu = \pm 2$ and $l^2 = a^4(k_2^2 + k_3^2)$.

For large u the UV boundary condition requires $C_2 = 0$. Therefore, in this region

$$\phi = u^{-2} C_1 K_2(lu) \tag{31}$$

For $u \ll \frac{1}{a}$ the equation simplifies to

$$\frac{1}{u^5}\partial_u u^5 \partial_u \phi - p^2(\frac{1}{u^4})\phi \quad = \quad 0 \tag{32}$$

where we have defined $p^2 = k_1^2 + k_2^2 + k_3^2$. The solution is found to be

$$\phi = u^{-2}(C_5 K_2(pu^{-1}) + C_6 I_2(pu^{-1})) \tag{33}$$

The near and far solutions should match in the intermediate region $u \sim a$. This then imposes the relations

$$C_5 = 0 \qquad C_6 = 16 C_1 p^{-2} l^{-2} \tag{34}$$

The approximate scalar field profile is then

$$\phi_+ = \left\{ \begin{array}{ll} 16u^{-2}I_2(pu^{-1}) & u \ll a^{-1} \\ u^{-2}p^2l^2K_2(lu) & u \gg a^{-1} \end{array} \right. \tag{35}$$

We have defined $p^2 = k_1^2 + k_2^2 + k_3^2$, $l^2 = a^4(k_2^2 + k_3^2)$.

Note that the solution for $u \ll a^{-1}$ is symmetric in all the directions, as it should be, since the background solution approaches the AdS_5 geometry

in this limit. On the other hand, the solution for $u \gg a^{-1}$ is not spherically symmetric.

Finally, we can now use these approximate solutions to find the shape of the resulting black holes. Let us first assume $au_c \ll 1$. We should then use the form of ϕ_+ for $u \ll a^{-1}$. The integral equation is then

$$\frac{M}{u_c^5} \int d^3k \ e^{ik_1x^1+ik_2x^2+ik_3x^3} \frac{u_c^{-2} I_2(pu_c^{-1})}{\partial_u (u^{-2} I_2(pu^{-1}))|_{u=u_c}} = 1 \tag{36}$$

This is exactly the same integral that one obtains for the AdS case. This is to be expected, since the geometry looks like AdS for $au_c \ll 1$. As in that case, the integral can be evaluated by contour integration and is dominated by the poles in the denominator. The dominant pole will appear at $p = \imath M_1$, where M_1 is determined by the precise form of the denominator (the pole must be imaginary, because the graviton solution is real for real momenta). One then finds $h_{00} \propto \frac{e^{-M_1 r}}{r}$. We therefore get the same results as before (i.e., $\sigma \sim \frac{1}{m_\pi^2} \ln^2 \frac{E}{m_\pi^2}$).

More concretely, we can expand the numerator near zero to get

$$e^{-\sqrt{u_c^2-k^2}x^i} e^{-kx} M u^{-2} R^2 a^4 k^2 (a^2 k u_c)^{-2} = e^{-\sqrt{u_c^2-k^2}x^i} e^{-kx} M R^2 (u_c)^{-4} \tag{37}$$

The numerator is spherically symmetric, so we get a spherical black hole whose size is independent of a. This is to be expected; for small a, the effects of noncommutativity are pushed out to very large u, so the black hole doesn't feel the noncommutativity. It doesn't matter how large the black hole gets, since that only modifies the shape far out, not near the brane.

Now take $au_c \gg 1$. We should then use the form of ϕ_+ for $u \gg a^{-1}$. The integral equation is then

$$\frac{M}{u_c^5} \int d^3k \ e^{ik_1x^1+ik_2x^2+ik_3x^3} \frac{u_c^{-2} K_2(lu_c)}{\partial_u (u^{-2} K_2(lu))|_{u=u_c}} = 1 \tag{38}$$

The integral over k_1 is now trivial (since $l^2 \equiv a^4(k_2^2 + k_3^2)$), and we get a $\delta(x_1)$ factor. The remaining integral can be performed as before (by contour integration, which is dominated by the pole), and we find a logarithmic growth in the x_2, x_3 directions.

We expect the delta function to be smoothed out by subleading terms. Indeed, this form of the far solution is not valid when k_1 is large.[3] If the

[3] equivalently, it is valid if $\frac{r_2^2+r_3^2}{r_1^2} \ll (au)^4$

integrand has support for large k_1, then this corresponds to a small but non-zero size for the black hole in the x_1 direction.

Note that the precise form of the denominator, as in the original AdS_5 case, is not very important. This is natural, as the Froissart bound should be a very general result. Indeed, we have only implemented the IR cutoff in a heuristic manner, modeled by cutting off the space with an IR brane. The fact that we can nevertheless see the Froissart-Martin bound in the AdS_5 case is an indication of how resilient this result is to changes in boundary conditions.

For this simple toy model, we thus get two results: if the infrared cutoff is taken to be much smaller than $\frac{1}{a}$, the shape of the black hole is very similar to the pure AdS case. On the other hand, if the infrared cutoff is taken to be very large, the shape of the black hole is very different – it is much smaller in the commutative directions, and still grows logarithmically in the noncommutative directions.

4 Scaling in the Commutative Directions

Our analysis of the scalar fluctuations has determined the parametric dependence of the size of the cross-section in the non-commutative directions. We have found that the size in the commutative directions is much smaller by comparison (when $au_c \gg 1$), but we would like to determine its behavior more precisely.

Now, if the cutoff u_c satisfies $u_c a \ll 1$, we should look at the behavior of ψ_+ near $u \sim 0$. In this region, the background geometry reduces to the $AdS_5 \times S^5$ geometry, and the behavior of the fluctuations will be the same as the AdS behavior. So the shape of the black hole in this limit will be the same as the shape found in [28].

The interesting case is therefore $u_c a \gg 1$, where the noncommutativity is important. In this region, the geometry again simplifies. In particular, the metric becomes

$$ds^2 = \alpha' R^2 \left[u^2(-dx_0^2 + dx_1^2) + \frac{1}{a^4 u^2}(dx_2^2 + dx_3^2) + \frac{du^2}{u^2} + d\Omega_5^2 \right] \tag{39}$$

Now define $u = u_c e^{\sigma}, \tilde{x}_0 = u_c x_0, \tilde{x}_1 = u_c x_1, \tilde{x}_2 = \frac{1}{a^2 u_c} x_2, \tilde{x}_3 = \frac{1}{a^2 u_c} x_3$. Then the metric becomes

$$ds^2 = \alpha' R^2 \left[e^{2\sigma}(-d\tilde{x}_0^2 + d\tilde{x}_1^2) + e^{-2\sigma}(d\tilde{x}_2^2 + d\tilde{x}_3^2) + d\sigma^2 + d\Omega_5^2 \right] \tag{40}$$

independent of a, u_c. The IR cutoff brane (where the source is located) is at $\sigma = 0$, which is also independent of a, u_c.

In this geometry, all the coefficients are of order 1, so we expect the size of the black hole to be parametrically the same in all directions. Furthermore, the calculations of the previous sections go through, and we find that the size should scale as $\ln m$ in the non-commutative directions. Thus, they should should scale as $\ln m$ in all the directions.

Rescaling back to the original coordinates, we find that the black hole radii should be

$$r_1 \sim \frac{\ln m}{u_c} \qquad r_2 \sim r_3 \sim (\ln m) a^2 u_c \tag{41}$$

which is consistent with our earlier result for the relative size of the cross-section in commutative and non-commutative directions.

5 Discussion

We have found that the scattering cross-section in NCFTs still obeys a Froissart-like bound $\sigma \propto (\ln s)^2$ at high energies. The interesting new feature is that the growth is not spherically symmetric; the size of the cross-section in the commutative directions grows more slowly than the size in the non-commutative directions[4]. Furthermore, this ratio of cross-sections is universal,

$$\frac{r_{2,3}}{r_1} \sim (a u_c)^2 \tag{42}$$

and depends only on the non-commutativity parameter a and the infrared cutoff u_c. Most importantly, this aspect ratio is independent of the energy of the scattering process.

This provides an answer to the question raised in the introduction: NCFT scattering at high energies is *not* dominated by the production of non-commutative solitons (if it were, then we would instead expect $\frac{r_{2,3}}{r_1} \sim E$). In this regard at least, NCFTs lack a crucial feature of quantum gravity. Note that the absence of linear scaling in the non-commutative directions also indicates that the behavior of the cross-section is not dominated by the stretched string

[4]Related work is the calculation of quantum commutators[36], from which one can also study fall-off behavior.

effect. One might wonder if this effect is somehow limited in the full non-perturbative theory at very high energies.[5]

The size of r_1 is indeed as expected. If we equate u_c with the mass gap m_π, then the scaling $r_1 \sim \frac{1}{m_\pi}\ln(\frac{s}{m_\pi^2})$ is the usual Froissart bound. The coefficient $\frac{1}{m_\pi}$ is heuristically related to idea that at the largest distance at which two particles interact, the potential energy between them should be approximately the mass of one quantum of the particle mediating the effective interaction.

For the $r_{2,3}$ directions it seems that the effective mass gap is given by $m_{\pi'} = \frac{1}{a^2 u_c}$, which will be much smaller than m_π in the limit $au_c \gg 1$. Indeed, one might have expected that the effective mass gap in the non-commutative directions should be smaller because of the additional poles of the form $p\theta^2 p$ which arise in NCFT from UV/IR mixing. However, the presence of poles at zero momentum would seem to suggest that the effective mass gap should be zero, and Froissart behavior should not appear at all. Instead, it appears that the strong coupling dynamics of the theory serves to "soften" the new poles, leaving a small but nonzero effective mass gap. The fact that the effective mass gap scales inversely with the putative mass gap may be an intriguing signature of the effective mass gap's origin in UV/IR mixing.

It may also be possible to interpret this scaling as the effect of a renormalization group flow. In this interpretation, the couplings at high energies are different in the commutative and noncommutative directions, and flow in the infrared to the usual NCFT action. The ratio of the values of the coupling constants for the two directions is given by $u_c a$. It would be interesting to see if this can be made more precise. Recent work [35] has also emphasized the non-intuitive behavior of NCFT's in the non-commutative directions. It would also be very interesting to understand the connection between our result and the reduction in the number of degrees of freedom in the high temperature limit of NCFT's found in [11, 12, 13, 37].

Our work here can be generalized to other cases. Noncommutative theories appear on the worldvolume of D-branes in curved spaces with background H-fields [38]. It would be interesting to see if the dual supergravity solution for these branes [39] exhibits a behavior similar to that found here, which would confirm that the Froissart behavior is universal.

Finally, we have used the scalar field as a toy model for the fully coupled

[5] We thank R. McNees for a discussion on this point.

perturbations of the metric and gauge fields. It would be very interesting (though extremely difficult) to verify our qualitative picture with a complete non-linear solution to the fully coupled equations.

Acknowledgments

We are grateful to C. Bauer, S. Cullen, J. Davis, M. Douglas, F. Larsen, B. Grinstein, R. Kaiser, J. Kuti, A. Manohar, R. McNees and especially S. Giddings, K. Intriligator and H. Nastase for useful discussions. J. K. would like to thank the University of California, San Diego, where work on this project was initiated. J. K. is supported by the Michigan Center for Theoretical Physics and the Department of Energy. A. R. is supported in part by NSF Grant 0354993.

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